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AN IMPROVED BOUNDARY FORCE METHOD FOR ANALYZING CRACKED ANISOTROPIC MATERIALS

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SUMMARY

In this paper, the Boundary Force Method (BFM), a form of an indirect boundary element method, is used to analyze composite laminates with cracks. The BFM uses the orthotropic elasticity solution for a concentrated horizontal and vertical force and a moment applied at a point in a cracked, infinite sheet as the fundamental solution. The necessary stress functions for this fundamental solution were formulated using the complex variable theory of orthotropic elasticity. The current method is an improvement over a previous method that used only forces and no moment. The improved method was verified by comparing it to accepted solutions for a finite-width, center-crack specimen subjected to uniaxial tension. Four graphite/epoxy laminates were used: $[0/\pm45/90]_s$, $[0]$, $[\pm45]_s$, and $[\pm30]_s$. The BFM results agreed well with accepted solutions. Convergence studies showed that with the addition of the moment in the fundamental solution, the number of boundary elements required for a converged solution was significantly reduced. Parametric studies were done for two configurations for which no orthotropic solutions are currently available: a single edge crack and an inclined single edge crack.

INTRODUCTION

Previous work for orthotropic materials (ref. 1) used the stress functions for a concentrated horizontal and vertical force applied at a point in a cracked, infinite sheet as the fundamental solution. The present work extends the fundamental solution to include a moment, as in the method for isotropic materials. The necessary stress functions for this fundamental solution are formulated using the complex variable theory of orthotropic elasticity (ref. 2). The addition of the moment DOF to the orthotropic formulation of the BFM is evaluated by comparing the BFM results to accepted solutions for a finite-width center-crack specimen subjected to uniaxial tension. Convergence studies are done comparing the methods for forces only and for moments and forces. Parametric studies are done for specimens with a single edge crack and an inclined single edge crack.

NOMENCLATURE

a	half length of a center crack or length of an edge crack, m
A, B, A_M, B_M	constants in stress functions, N/m^3
C_{ij}	complex constants ($i,j = 1,2$)
E_x, E_y	Young's moduli in the x- and y-directions, respectively, Pa
F_I, F_{II}	mode I and mode II stress-intensity correction factors
$[F]$	influence coefficient matrix, N/m
G_{xy}	orthotropic shear modulus, Pa
H	height of plate, m
K_I, K_{II}	mode I and mode II stress-intensity factors, Pa$\sqrt{m}$
m_i	unit moment on the i^{th} boundary element, N-m
M	concentrated moment, N-m

N	number of boundary elements
N(x), T(x)	normal and shear crack-face loading functions, Pa
p_i, q_i	units loads on the i^{th} boundary element, N
P, Q	concentrated forces in the x- and y-directions, respectively, N
{P}	vector of unknown forces, N
R_x, R_y	x- and y-components of applied loading, N
{R}	vector of external loads, N
S	remote applied stress, Pa
t	location of load point on crack face, m
T	externally applied moment, N-m
w_i	complex variable, (i = 1 to 4)
W	width or half-width of plate, m
x, y	Cartesian coordinates, m
z_0	load point, $(z_0 = x_0 + iy_0)$, m
δ_i	incremental distance, (i = 1 to N), m
$\phi_1'(z)$, $\phi_2'(z)$	orthotropic stress functions, Pa
ν_{xy}, ν_{yx}	Poisson's ratios
σ_x, σ_y, τ_{xy}	stresses, Pa
$\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2$	roots of the characteristic equation

BOUNDARY FORCE METHOD

The Boundary Force Method (ref. 3) is a numerical technique which uses fundamental solutions for concentrated forces and moments in an infinite sheet to obtain the solution to the boundary value problem of interest. These fundamental solutions are used to exactly satisfy the stress-free conditions on the crack faces. The other boundary conditions are approximately satisfied by applying the appropriate sets of concentrated horizontal and vertical forces and moments along the boundary.

Consider, for example, the finite cracked plate subjected to uniaxial tension shown in Figure 1. In the BFM, an imaginary boundary corresponding to the finite plate is traced on a cracked infinite sheet. These boundaries are then divided into a finite number of boundary elements. On each boundary element, a concentrated force pair P_i and Q_i and a moment M_i ($i = 1$ to N) are applied at a small distance δ_i on the outward normal from the mid-point of the boundary element. This small offset from the boundary element was used to avoid the inconvenience of singularities in the computation of the stresses on the boundaries. In the present work, δ_i was set equal to one quarter of the element length.

The concentrated forces P_i and Q_i and moment M_i acting on the i^{th} boundary element produce stresses on the other boundary elements. The resultant forces and moments on each boundary element j are found by integrating the stresses over the boundary element length (assuming a unit thickness). On each boundary element j, the sum of all the resultant forces and moments must be equal to the externally applied forces and moments acting on the boundary element. Denoting the externally applied forces as R_{x_j} and R_{y_j}, and the externally applied moment as T_j, we can write the following equilibrium equations for the j^{th} boundary element:

$$R_{x_j} = \sum_{i=1}^{N} (F_{x_j p_i} P_i + F_{x_j q_i} Q_i + F_{x_j m_i} M_i)$$

$$R_{y_j} = \sum_{i=1}^{N} (F_{y_j p_i} P_i + F_{y_j q_i} Q_i + F_{y_j m_i} M_i) \tag{1}$$

$$T_j = \sum_{i=1}^{N} (C_{jp_i} P_i + C_{jq_i} Q_i + C_{jm_i} M_i)$$

Here N is the total number of boundary elements; $F_{x_j p_i}$, $F_{y_j p_i}$, $F_{x_j q_i}$, $F_{y_j q_i}$, $F_{x_j m_i}$, $F_{y_j m_i}$, C_{jp_i}, C_{jq_i}, and C_{jm_i} are called influence coefficients and are defined as follows.

$F_{x_j p_i}$ = force per unit force in the x-direction on the j^{th} boundary element due to unit load p_i acting in the y-direction on the i^{th} boundary element.

$F_{y_j p_i}$ = force per unit force in the y-direction on the j^{th} boundary element due to unit load p_i acting in the y-direction on the i^{th} boundary element.

C_{jp_i} = resultant couple per unit force created on the j^{th} boundary element due to unit load p_i acting in the y-direction on the i^{th} boundary element.

$F_{x_j q_i}$, $F_{y_j q_i}$, $F_{x_j m_i}$, $F_{y_j m_i}$, C_{jq_i}, and C_{jm_i} are defined in a similar manner.

The resulting system of equations for N boundary elements can be written as

$$[F]_{3N\times3N}\,\{P\}_{3N\times1} = \{R\}_{3N\times1} \tag{2}$$

where $[F]$ is the influence coefficient matrix, $\{P\}$ is the vector of unknowns, P, Q, and M, and $\{R\}$ is the vector of externally applied forces. The influence coefficient matrix is square, fully populated and non-symmetric.

Because the influence coefficient matrix and the externally applied load vector are known, the unknown force vector can be obtained by solving the system of linear algebraic simultaneous equations. The calculated set of P_i, Q_i, and M_i acting on the imaginary boundaries in the sheet will approximately satisfy the required boundary conditions, and, thus, produce a stress distribution inside the imaginary boundaries that is approximately equal to the stress distribution of the desired boundary value problem.

DERIVATION OF FUNDAMENTAL SOLUTION FOR ORTHOTROPIC MATERIALS

For orthotropic materials, the BFM uses the elasticity solution for a horizontal and vertical concentrated force and a concentrated moment in an infinite orthotropic sheet with a crack. The formulation of this solution is presented below.

Stresses in Orthotropic Materials

From Lekhnitskii (ref. 2), the stresses in an infinite orthotropic sheet can be written in terms of the two stress functions, ϕ_1' and ϕ_2', as follows:

$$\sigma_x(x,y) = 2\,\mathrm{Re}\left[\mu_1^2\phi_1'(z_1) + \mu_2^2\phi_2'(z_2)\right]$$

$$\sigma_y(x,y) = 2\,\mathrm{Re}\left[\phi_1'(z_1) + \phi_2'(z_2)\right] \tag{3}$$

$$\tau_{xy}(x,y) = -2\,\mathrm{Re}\left[\mu_1\phi_1'(z_1) + \mu_2\phi_2'(z_2)\right]$$

For simplicity, the derivation of the stress functions for the concentrated forces and for the moment acting at an arbitrary point in a infinite, cracked, orthotropic sheet will be presented separately.

Concentrated Forces. The derivation of the stress functions for a pair of concentrated forces acting on a cracked, infinite orthotropic sheet was presented in ref. 1. For completeness, the derivation of these stress functions is shown in Appendix A. The stress functions are given below.

$$\phi_1'(z_1) = \frac{1}{2(\mu_1 - \mu_2)\sqrt{z_1^2 - a^2}}\left[\frac{(\mu_2 - \mu_1)A}{z_1 - w_1} f(z_1, w_1, a) + \frac{(\mu_2 - \bar{\mu}_1)\bar{A}}{z_1 - \bar{w}_1} f(z_1, \bar{w}_1, a) + \frac{(\mu_2 - \bar{\mu}_2)\bar{B}}{z_1 - \bar{w}_2} f(z_1, \bar{w}_2, a)\right] + \frac{A}{z_1 - w_1}$$

(4)

$$\phi_2'(z_2) = \frac{-1}{2(\mu_1 - \mu_2)\sqrt{z_2^2 - a^2}}\left[\frac{(\mu_1 - \mu_2)B}{z_2 - w_2} f(z_2, w_2, a) + \frac{(\mu_1 - \bar{\mu}_1)\bar{A}}{z_2 - \bar{w}_1} f(z_2, \bar{w}_1, a) + \frac{(\mu_1 - \bar{\mu}_2)\bar{B}}{z_2 - \bar{w}_2} f(z_2, \bar{w}_2, a)\right] + \frac{B}{z_2 - w_2}$$

where $f(z, w, a) = \sqrt{z^2 - a^2} - \sqrt{w^2 - a^2} - z + w$, and the other terms in equations (4) are defined in Appendix A.

Moment. Figure 2 shows how superposition was used to determine the stress functions for a moment acting on a cracked, infinite orthotropic sheet. The uncracked sheet with the moment applied at z_0 is shown in Figure 2(b). The normal and shear stresses acting on the line $y = 0$, $|x| < a$, are shown as $N(x)$ and $T(x)$. The problem with the crack-face loading (Figure 2(c)) is superimposed on the uncracked problem (Figure 2(b)) to produce the stress-free crack face shown in Figure 2(a).

The stress functions for a moment in an infinite, uncracked orthotropic sheet (used for Figure 2(b)) are derived (following ref. 4) in Appendix B. These stress functions are

$$\phi_1'(z_1) = \frac{-A_M}{(z_1 - w_1)^2} \qquad \phi_2'(z_2) = \frac{-B_M}{(z_2 - w_2)^2} \tag{5}$$

where

$$A_M = \frac{(\mu_1 C_{11} - C_{12})M}{2} \qquad B_M = \frac{(\mu_2 C_{21} - C_{22})M}{2}$$

The terms μ_1, μ_2, C_{11}, C_{12}, C_{21}, and C_{22} are defined in Appendix A.

The stress functions for the loads $N(x)$ and $T(x)$ applied to the crack face in Figure 2(c) are given in Appendix A, equations (A4). Here,

$N(x) = \sigma_y(x,0)$ and $T(x) = \tau_{xy}(x,0)$ in the uncracked sheet in Figure 2(b). From equations (3) and (5),

$$\sigma_y(x,0) = -2\ \mathrm{Re}\left[\frac{A_M}{(z_1 - w_1)^2} + \frac{B_M}{(z_2 - w_2)^2}\right]$$

$$\tau_y(x,0) = 2\ \mathrm{Re}\left[\frac{\mu_1 A_M}{(z_1 - w_1)^2} + \frac{\mu_2 B_M}{(z_2 - w_2)^2}\right] \tag{6}$$

Using equations (6) and remembering that $2\ \mathrm{Re}[f(z)] = f(z) + \overline{f(z)}$, the loading functions in the integrands in equations (A4) can be simplified as follows:

$$\mu_2 N(x) + T(x) = -2\ \mu_2 \mathrm{Re}\ [\frac{A_M}{(x - w_1)^2} + \frac{B_M}{(x - w_2)^2}] + 2\ \mathrm{Re}\ [\frac{\mu_1 A_M}{(x - w_1)^2} + \frac{\mu_2 B_M}{(x - w_2)^2}]$$

$$= -\frac{(\mu_2 - \mu_1)A_M}{(x - w_1)^2} - \frac{(\mu_2 - \bar{\mu}_1)\bar{A}_M}{(x - \bar{w}_1)^2} - \frac{(\mu_2 - \bar{\mu}_2)\bar{B}_M}{(x - \bar{w}_2)^2}$$

$$\tag{7}$$

$$\mu_1 N(x) + T(x) = -2\ \mu_1 \mathrm{Re}\ [\frac{A_M}{(x - w_1)^2} + \frac{B_M}{(x - w_2)^2}] + 2\ \mathrm{Re}\ [\frac{\mu_1 A_M}{(x - w_1)^2} + \frac{\mu_2 B_M}{(x - w_2)^2}]$$

$$= -\frac{(\mu_1 - \bar{\mu}_1)\bar{A}_M}{(x - \bar{w}_1)^2} - \frac{(\mu_1 - \mu_2)B_M}{(x - w_2)^2} - \frac{(\mu_1 - \bar{\mu}_2)\bar{B}_M}{(x - \bar{w}_2)^2}$$

Substituting these expressions in equations (A4), integrating, and then adding equations (5), the following expressions are found for the stress functions for the loading shown in Figure 2(a):

$$\phi_1'(z_1) = \frac{-1}{2(\mu_1 - \mu_2)\sqrt{z_1^2 - a^2}} [(\mu_2 - \mu_1) A_M D(z_1,w_1,a)$$
$$+ (\mu_2 - \bar{\mu}_1) \bar{A}_M D(z_1,\bar{w}_1,a) + (\mu_2 - \bar{\mu}_2) \bar{B}_M D(z_1,\bar{w}_2,a)]$$
$$+ \frac{-A_M}{(z_1 - w_1)^2}$$

$$\phi_2'(z_2) = \frac{1}{2(\mu_1 - \mu_2)\sqrt{z_2^2 - a^2}} [(\mu_1 - \mu_2) B_M D(z_2,w_2,a) \tag{8}$$
$$+ (\mu_1 - \bar{\mu}_1) \bar{A}_M D(z_2,\bar{w}_1,a) + (\mu_1 - \bar{\mu}_2) \bar{B}_M D(z_2,\bar{w}_2,a)]$$
$$+ \frac{-B_M}{(z_2 - w_2)^2}$$

where

$$D(z,w,a) = \frac{(w^2 - a^2)\sqrt{z^2 - a^2} + (a^2 - wz)\sqrt{w^2 - a^2}}{(w^2 - a^2)(z - w)^2}$$

The stress functions used in the fundamental solution for orthotropic materials are found by combining equations (4) and (8). Once the stress functions are known, it is a simple step to calculate the stresses at any point in the body using equations (3). Then, as mentioned earlier, the stresses are integrated over each element length to obtain resultant forces for use in equation (2).

Stress-Intensity Factor Equation

From Snyder and Cruse (ref. 5), the stress-intensity factors for orthotropic materials may be expressed in terms of the stress functions as follows:

$$K_I + \frac{K_{II}}{\mu_2} = -2\sqrt{2\pi}\,\frac{\mu_1 - \mu_2}{\mu_2} \lim_{z_1 \to a} \{ \sqrt{z_1 - a}\ \phi'(z_1) \} \tag{9}$$

By substituting from equations (4) and (8) into the above equation and taking the limit, the mode I and mode II components of the stress-intensity factor for a horizontal and vertical force and a moment in an infinite cracked orthotropic sheet may be written as follows:

$$\begin{aligned} K_I + \frac{K_{II}}{\mu_2} = \frac{1}{\mu_2}\sqrt{\pi/a}\ \{ & (\mu_2 - \mu_1)\ [A\ G(w_1,a) + A_M\ H(w_1,a)] \\ & + (\mu_2 - \bar{\mu}_1)\ [\bar{A}\ G(\bar{w}_1,a) + \bar{A}_M\ H(\bar{w}_1,a)] \\ & + (\mu_2 - \bar{\mu}_2)\ [\bar{B}\ G(\bar{w}_2,a) + \bar{B}_M\ H(\bar{w}_2,a)] \end{aligned} \tag{10}$$

where

$$G(w,a) = \frac{\sqrt{w^2 - a^2} - w + a}{w - a} \qquad H(w,a) = \frac{a\sqrt{w^2 - a^2}}{(w^2 - a^2)(a - w)}$$

RESULTS AND DISCUSSION

The following four graphite/epoxy (gr/ep) laminates were used in the analysis: $[0/\pm45/90]_s$, $[0]$, $[\pm45]_s$, and $[0/\pm45]_s$. Table 1 presents the laminate constants for the four laminates. The 0^o lamina properties (ref. 5) were used with lamination theory to calculate the elastic constants for the other laminates. (Here, the 0^o-direction is defined parallel to the load axis.) The results for the $[0/\pm45/90]_s$ quasi-isotropic laminate were compared with isotropic solutions from the literature. To show the effects of the specimen boundaries, the results are presented using the following stress-intensity correction factors F_I and F_{II}:

$$K_I = S\sqrt{\pi a}\, F_I \qquad\qquad K_{II} = S\sqrt{\pi a}\, F_{II}$$

Verification

To evaluate the improved Boundary Force Method for orthotropic materials, the BFM stress-intensity factor solutions were compared to results from Snyder and Cruse (ref. 5) for a center-crack tension specimen with a finite width of $H/W = 3.0$. In Figure 3, the curves represent the BFM calculations, while the symbols indicate the values taken from Snyder and Cruse. The stress-intensity correction factors calculated by the BFM agree, within ±3%, with the values from Snyder and Cruse for all laminates considered. For the $[0/\pm45/90]_s$ laminate at $2a/W = 0.8$, the solution from Snyder and Cruse was 2% lower than the solution for an isotropic material (ref. 6), whereas the BFM solution for the $[0/\pm45/90]_s$ laminate was only in error by .005% compared to the accepted isotropic solution. Therefore, for the other laminates considered, the 3% difference between the present results and those of Snyder and Cruse may represent an error in the solution of Snyder and Cruse.

Convergence Studies

To demonstrate the benefit of the addition of the moment, a convergence study was done for a single edge crack in a quasi-isotropic plate. Figure 4 shows the convergence of the solution for an a/W ratio of 0.6. The solution with both forces and moments converged to within 1.5% of the solution in ref.

7 with 63 DOF; the solution with only forces required 117 DOF to converge to the reference solution.

Convergence studies such as that shown in Figure 4 were done for other laminates shown in Table 1 with an edge crack to determine if the material properties had any effect on the convergence rate. Although each converged to a different solution, the number of degrees of freedom required for convergence was similar for all the laminates. In general, with the moment included, between 80 to 100 DOF were required for convergence for configurations with deep cracks ($a/W \geq 0.6$). Fewer DOF are required for smaller a/W ratios.

Edge Crack Solutions

Stress-intensity correction factors were also calculated for a single edge crack and an inclined edge crack for a range of crack-length-to-width ratios and material properties. Other orthotropic solutions for these two problems were not available for comparison.

Single Edge Crack. Figure 5 shows the stress-intensity correction factor F for a single edge crack for four graphite/epoxy laminates: $[0/\pm45/90]_s$, $[0]$, $[\pm45]_s$, and $[90]$. The results for the quasi-isotropic laminate agreed within 0.005% with the isotropic results for this configuration. For very small a/W ratios, the solutions for the different laminates do not tend toward a single value, as is the case in the center crack configuration (Figure 3). Thus, the anisotropy of the material has more effect for small edge cracks than for small center-cracks. Table 2 lists correction factors plotted in Figure 5.

Inclined Edge Crack. Figures 6 and 7 show F_I and F_{II}, respectively, for an inclined edge crack for three graphite/epoxy laminates: $[0/\pm45/90]_s$, [0], and [90]. In Figure 6, the mode I component does not converge to a single value for small a/W ratios, much as in Figure 5. The mode II component shown in Figure 7 varies less with anisotropy than the mode I component shown in Figure 6. Tables 3 and 4 list the correction factors plotted in Figures 6 and 7.

CONCLUDING REMARKS

In this paper, the Boundary Force Method (BFM), a form of an indirect boundary element method, is used to analyze composite laminates with cracks. The BFM uses the orthotropic elasticity solution for a concentrated horizontal and vertical force and a moment applied at a point in a cracked, infinite sheet as the fundamental solution. This formulation is an improvement over a previous method that did not include the moment but used only the horizontal and vertical forces on the boundary. The necessary stress functions for this fundamental solution were derived using the complex variable theory of orthotropic elasticity. The orthotropic formulation of the BFM was verified by comparing solutions for a center-crack specimen subjected to uniaxial tension to other solutions. The BFM results agreed well with accepted solutions.

Parametric studies were also done for a single edge crack, and an inclined edge crack, both loaded in uniaxial tension, with a variety of materials properties. Only a slight effect was seen due to material anisotropy. No other orthotropic solutions were available for comparison.

The additions of the moment degree of freedom to the orthotropic BFM was shown to greatly increase the convergence rate of the solution. Thus, fewer boundary elements were required to achieve the same accuracy.

This work has resulted in a further extension of the Boundary Force Method in the analysis of composite laminates with cracks and notches. This method yields accurate solutions with minimal modeling effort, even for complex configurations. The accurate stress-intensity factors obtained with this method should be useful in predicting fracture strengths of arbitrarily shaped composite laminates.

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APPENDIX A - DERIVATION OF STRESS FUNCTIONS FOR POINT LOADS IN CRACKED, ORTHOTROPIC SHEET

This appendix (taken from ref. 1) presents the derivation of the stress function for a pair of point loads in a cracked, infinite orthotropic sheet. Figure 8 shows how superposition was used to determine the stress functions. The uncracked sheet with the point loads applied at z_0 and the stresses along $y = 0$, $|x| < a$, are shown in Figure 8(b). The point loads in Figure 8(b) are superimposed on the stresses due to the crack-face loadings in Figure 8(c) to produce the stress-free crack face shown in Figure 8(a).

The stress functions for a point load in an infinite, orthotropic sheet (used for Figure 8(b)) are derived from Lekhnitskii (ref. 2):

$$\phi_1'(z_1) = \frac{A}{(z_1 - w_1)} \qquad \phi_2'(z_2) = \frac{B}{(z_2 - w_2)} \tag{A1}$$

where

$$A = C_{11}P + C_{12}Q \qquad B = C_{21}P + C_{22}Q$$

$$w_j = x_0 + \mu_j y_0 \qquad z_j = x + \mu_j y \qquad (j = 1,2)$$

$$C_{11} = \frac{\mu_1 \, [\bar{\mu}_1(1 + \nu_{yx}\mu_2\bar{\mu}_2) + \bar{\mu}_2 + \mu_2]}{2\pi i(\mu_1 - \mu_2)(\mu_1 - \bar{\mu}_1)(\mu_1 - \bar{\mu}_2)}$$

$$C_{12} = \frac{\mu_1 \, [(\bar{\mu}_2 + \mu_2)\bar{\mu}_1 + \mu_2\bar{\mu}_2 + \nu_{xy}]}{2\pi i(\mu_1 - \mu_2)(\mu_1 - \bar{\mu}_1)(\mu_1 - \bar{\mu}_2)}$$

$$C_{21} = \frac{\mu_2 \, [(1 + \nu_{yx}\mu_1\bar{\mu}_1)\bar{\mu}_2 + \bar{\mu}_1 + \mu_1]}{2\pi i(\mu_2 - \mu_1)(\mu_2 - \bar{\mu}_2)(\mu_2 - \bar{\mu}_1)} \qquad \text{(A2)}$$

$$C_{22} = \frac{\mu_2 \, [(\bar{\mu}_1 + \mu_1)\bar{\mu}_2 + \mu_1\bar{\mu}_1 + \nu_{xy}]}{2\pi i(\mu_2 - \mu_1)(\mu_2 - \bar{\mu}_2)(\mu_2 - \bar{\mu}_1)}$$

Here ν_{xy} and ν_{yx} are the Poisson's ratios; μ_1, μ_2, $\bar{\mu}_1$, $\bar{\mu}_2$ are the complex roots of the characteristic equation $\left\{\mu^4 + (\frac{E_x}{G_{xy}} - 2\nu_{xy})\mu^2 + \frac{E_x}{E_y} = 0\right\}$; and the barred quantities represent the complex conjugates of the underlying functions.

For isotropic materials, the roots of the characteristic equation are $\mu_1 = \mu_2 = i$, and the term $(\mu_1 - \mu_2)$ in the denominator of equations (A2) is zero. When using the orthotropic equations for isotropic materials, a small perturbation was introduced in the values of μ_1 and μ_2 so that $\mu_1 = i(1 + \epsilon)$ and $\mu_2 = i(1 - \epsilon)$ where $\epsilon = 0.0001$.

To find the stress functions for the loading shown in Figure 8(c), the stress functions for a point load applied at an arbitrary point on the crack face are used. From Savin (ref. 8), the stress functions for this loading are as follows:

$$\phi_1'(z_1) = \frac{-\mu_2 P - Q}{2\pi(\mu_1 - \mu_2)} \frac{1}{\sqrt{z_1^2 - a^2}} \frac{\sqrt{a^2 - t^2}}{z_1 - t}$$

(A3)

$$\phi_2'(z_2) = \frac{\mu_1 P + Q}{2\pi(\mu_1 - \mu_2)} \frac{1}{\sqrt{z_2^2 - a^2}} \frac{\sqrt{a^2 - t^2}}{z_2 - t}$$

where P and Q are the normal and tangential point loads applied to the crack face, and t is the location of the load point on the crack face ($-a < t < a$), as shown in Figure 9.

By integrating equations (A3) over the crack face, the stress functions for the non-uniform distributed loads applied to the crack face can be written as follows:

$$\phi_1'(z_1) = \frac{-1}{2\pi(\mu_1 - \mu_2)} \frac{1}{\sqrt{z_1^2 - a^2}} \int_{-a}^{a} [\mu_2 N(\xi) + T(\xi)] \frac{\sqrt{a^2 - \xi^2}}{z_1 - \xi} d\xi$$

(A4)

$$\phi_2'(z_2) = \frac{1}{2\pi(\mu_1 - \mu_2)} \frac{1}{\sqrt{z_2^2 - a^2}} \int_{-a}^{a} [\mu_1 N(\xi) + T(\xi)] \frac{\sqrt{a^2 - \xi^2}}{z_2 - \xi} d\xi$$

In order to make the crack stress free along the crack line $-a < x < a$, the loadings $N(x)$ and $T(x)$ are specified to be the same as the stresses found in the equivalent uncracked sheet shown in Figure 8(b). That is, $N(x) = \sigma_y(x,0)$ and $T(x) = \tau_{xy}(x,0)$. Thus, from equations (3) and (A1), the normal and shear stresses on the line $y = 0$ are

$$\sigma_y(x,0) = 2 \text{ Re } \left(\frac{A}{x - w_1} + \frac{B}{x - w_2}\right)$$

$$\tau_{xy}(x,0) = -2 \text{ Re } \left(\frac{\mu_1 A}{x - w_1} + \frac{\mu_2 B}{x - w_2}\right) \tag{A5}$$

Using equations (A5) and remembering that $2 \text{ Re } [f(z)] = f(z) + \overline{f(z)}$, the loading functions in the integrands in equations (A4) can be simplified as follows:

$$\mu_2 N(x) + T(x) = 2\, \mu_2 \text{ Re } \left(\frac{A}{x - w_1} + \frac{B}{x - w_2}\right) - 2 \text{ Re } \left(\frac{\mu_1 A}{x - w_1} + \frac{\mu_2 B}{x - w_2}\right)$$

$$= \frac{(\mu_2 - \mu_1)A}{x - w_1} + \frac{(\mu_2 - \bar{\mu}_1)\bar{A}}{x - \bar{w}_1} + \frac{(\mu_2 - \bar{\mu}_2)\bar{B}}{x - \bar{w}_2}$$

(A6)

$$\mu_1 N(x) + T(x) = 2\, \mu_1 \text{ Re } \left(\frac{A}{x - w_1} + \frac{B}{x - w_2}\right) - 2 \text{ Re } \left(\frac{\mu_1 A}{x - w_1} + \frac{\mu_2 B}{x - w_2}\right)$$

$$= \frac{(\mu_1 - \mu_2)B}{x - w_2} + \frac{(\mu_1 - \bar{\mu}_1)\bar{A}}{x - \bar{w}_1} + \frac{(\mu_1 - \bar{\mu}_2)\bar{B}}{x - \bar{w}_2}$$

Substituting these expressions into equations (A4), integrating, and then adding equations (A1), the following expressions are found for the stress functions for the loading shown in Figure 8(a):

$$\phi_1'(z_1) = \frac{1}{2(\mu_1 - \mu_2)\sqrt{z_1^2 - a^2}} \left[\frac{(\mu_2 - \mu_1)A}{z_1 - w_1} f(z_1, w_1, a) + \frac{(\mu_2 - \bar{\mu}_1)\bar{A}}{z_1 - \bar{w}_1} f(z_1, \bar{w}_1, a) \right.$$

$$\left. + \frac{(\mu_2 - \bar{\mu}_2)\bar{B}}{z_1 - \bar{w}_2} f(z_1, \bar{w}_2, a) \right] + \frac{A}{z_1 - w_1}$$

(A7)

$$\phi_2'(z_2) = \frac{-1}{2(\mu_1 - \mu_2)\sqrt{z_2^2 - a^2}} \left[\frac{(\mu_1 - \mu_2)B}{z_2 - w_2} f(z_2, w_2, a) + \frac{(\mu_1 - \bar{\mu}_1)\bar{A}}{z_2 - \bar{w}_1} f(z_2, \bar{w}_1, a) \right.$$

$$\left. + \frac{(\mu_1 - \bar{\mu}_2)\bar{B}}{z_2 - \bar{w}_2} f(z_2, \bar{w}_2, a) \right] + \frac{B}{z_2 - w_2}$$

where $f(z, w, a) = \sqrt{z^2 - a^2} - \sqrt{w^2 - a^2} - z + w$.

(These stress functions are identical to those presented by Snyder and Cruse (ref. 5), derived by formulating the problem as a Hilbert problem.)

APPENDIX B - DERIVATION OF STRESS FUNCTIONS FOR MOMENT IN INFINITE, UNCRACKED ORTHOTROPIC SHEET

This appendix presents the derivation of the stress functions for a moment applied in an infinite uncracked orthotropic sheet. This derivation follows the procedure outlined in ref. 4.

Consider the pair of forces applied as shown in Figure 10(a). By superimposing the stress functions for concentrated forces in the y-direction, the stress functions for this loading are

$$\phi_1(z_1) = C_{12}P_y\log(z_1 - w_1) - C_{12}P_y\log(z_1 - w_1 - \epsilon)$$

$$\phi_2(z_2) = C_{22}P_y\log(z_2 - w_2) - C_{22}P_y\log(z_2 - w_2 - \epsilon) \tag{B2}$$

To obtain the stress functions for a moment due to a force pair in the y-direction, set $M = -P_y\epsilon$ and take the limit of equations (B2) as ϵ approaches zero.

$$\phi_1(z_1) = \frac{-C_{12}M}{z_1 - w_1} \qquad \phi_2(z_2) = \frac{-C_{22}M}{z_2 - w_2} \tag{B3}$$

Consider the pair of forces applied as shown in Figure 10(b). By superimposing the stress functions for concentrated forces in the x-direction, the stress functions for this loading are

$$\phi_1(z_1) = C_{11}P_x\log(z_1 - w_1) - C_{11}P_x\log(z_1 - w_1 + \mu_1\epsilon)$$

$$\tag{B4}$$

$$\phi_2(z_2) = C_{21}P_x\log(z_2 - w_2) - C_{21}P_x\log(z_2 - w_2 + \mu_2\epsilon_y)$$

To obtain the stress functions for a double force due to a moment in the x-direction, set $M = -P_x\epsilon$ and take the limit of equations (B4) as ϵ approaches zero.

$$\phi_1(z_1) = \frac{\mu_1 C_{11} M}{z_1 - w_1} \qquad \phi_2(z_2) = \frac{\mu_2 C_{21} M}{z_2 - w_2} \tag{B5}$$

By superimposing these stress functions, the solution for a concentrated moment due to the two force pairs ($M = -2P\epsilon$) can be written as

$$\phi_1(z_1) = \frac{(\mu_1 C_{11} - C_{12})M}{2(z_1 - w_1)} \qquad \phi_2(z_2) = \frac{(\mu_2 C_{21} - C_{22})M}{2(z_2 - w_2)} \tag{B6}$$

$$\phi_1'(z_1) = \frac{-(\mu_1 C_{11} - C_{12})M}{2(z_1 - w_1)^2} \qquad \phi_2'(z_2) = \frac{-(\mu_2 C_{21} - C_{22})M}{2(z_2 - w_2)^2} \tag{B7}$$

Table 1 - Laminate Constants

Laminate	E_x, GPa	E_y, GPa	ν_{xy}	G_{xy}, GPa
$[0/\pm45/90]_s$	60.04	60.04	0.259	23.85
[0]	11.72	144.8	0.017	9.65
$[\pm45]_s$	31.18	31.18	0.615	38.03
$[\pm30]_s$	17.64	66.99	0.262	30.94
[90]	144.8	11.72	0.210	9.65

Table 2 - Stress-Intensity Correction Factors for Edge Crack

a/w	F			
	$[0/\pm45/90]_s$	[0]	$[\pm45]_s$	[90]
.1	1.1894	1.1589	1.2861	1.1889
.2	1.3656	1.3576	1.4537	1.3600
.3	1.6567	1.6923	1.7623	1.6397
.4	2.1062	2.1884	2.2088	2.0594
.5	2.8015	2.9034	3.0347	2.9772
.6	4.0021	4.2344	4.2323	3.7939

Table 3 - Stress-Intensity Correction Factors for Inclined Edge Crack

a/w	F_I		
	$[0/\pm45/90]_s$	[0]	[90]
.1	0.7113	0.7495	0.7164
.2	0.8130	0.8180	0.8043
.3	0.8713	0.8987	0.8759
.4	1.0091	1.0407	1.0113
.5	1.1952	1.2095	1.1969
.6	1.4442	1.4792	1.4462

Table 4 - Stress-Intensity Correction Factors for Inclined Edge Crack

a/w	F_{II}		
	$[0/\pm 45/90]_s$	[0]	[90]
.1	0.3739	0.3737	0.3760
.2	0.3957	0.4029	0.4001
.3	0.4452	0.4451	0.4472
.4	0.5024	0.5026	0.5047
.5	0.5795	0.5804	0.5815
.6	0.6804	0.6855	0.6820

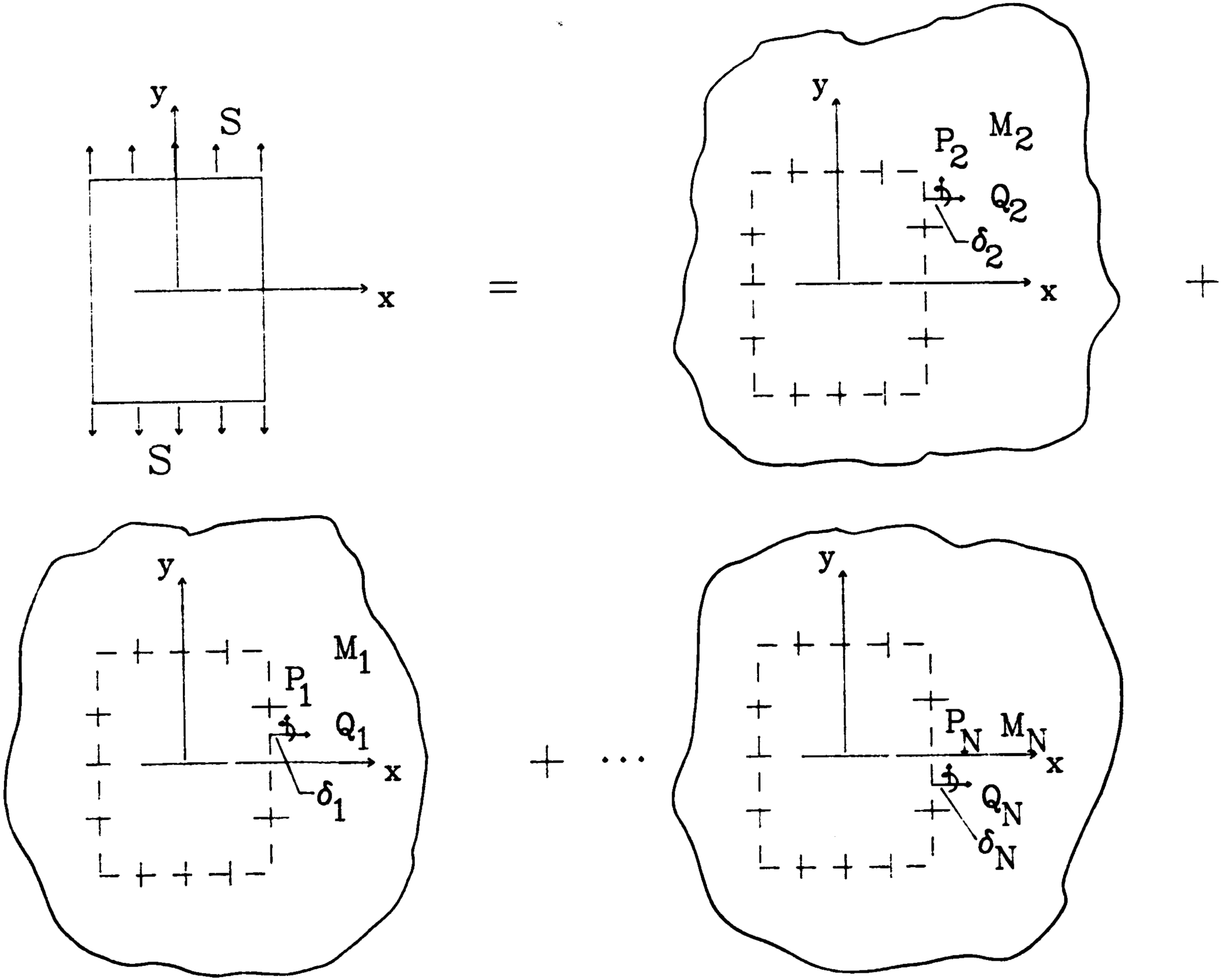

Figure 1. Superposition for Boundary Force Method.

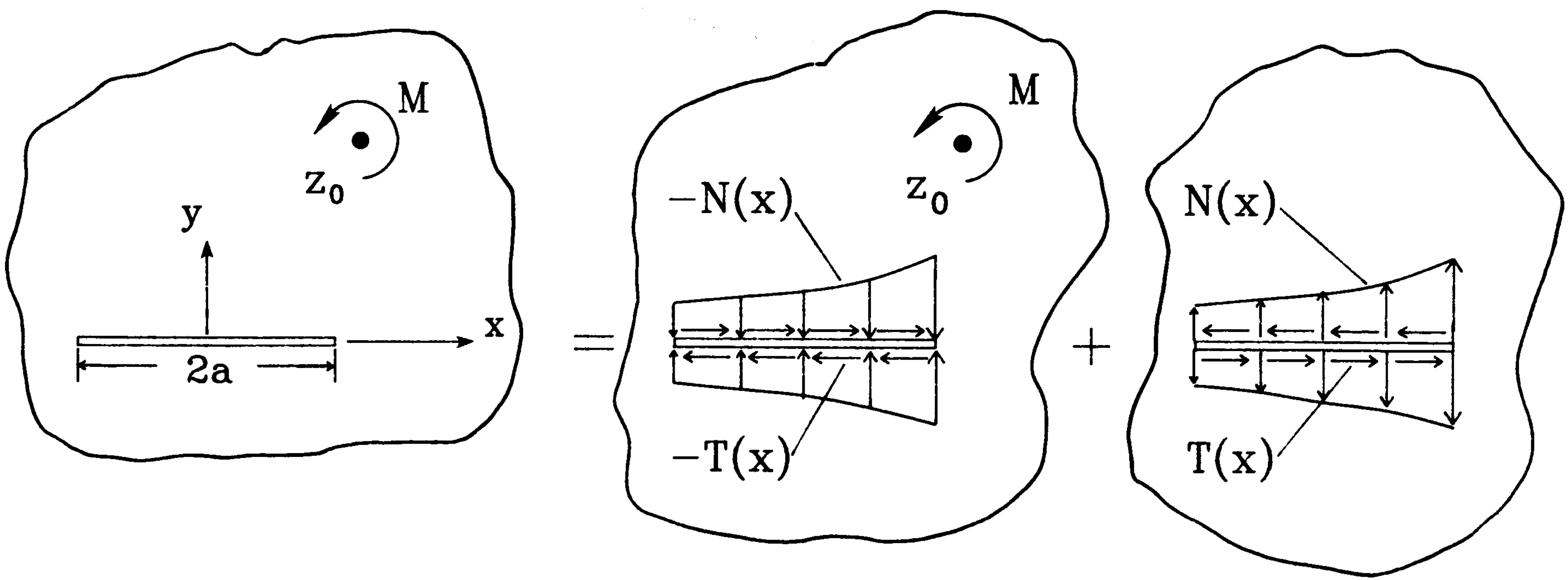

Figure 2. Superposition for formulation of stress functions for moment in a cracked, orthotropic sheet.

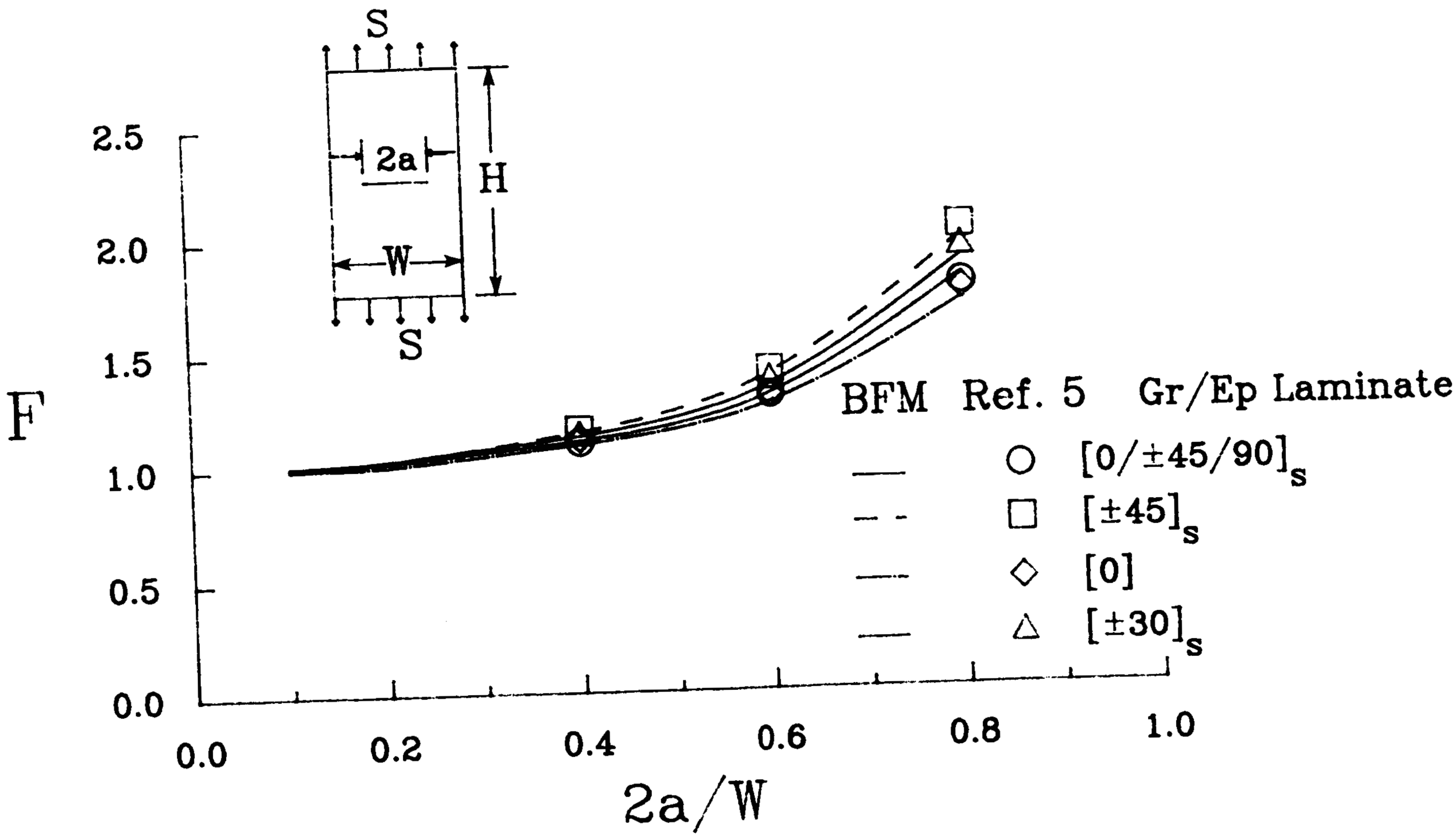

Figure 3. Stress-intensity correction factors for a center-crack tension specimen. H/W = 3.0.

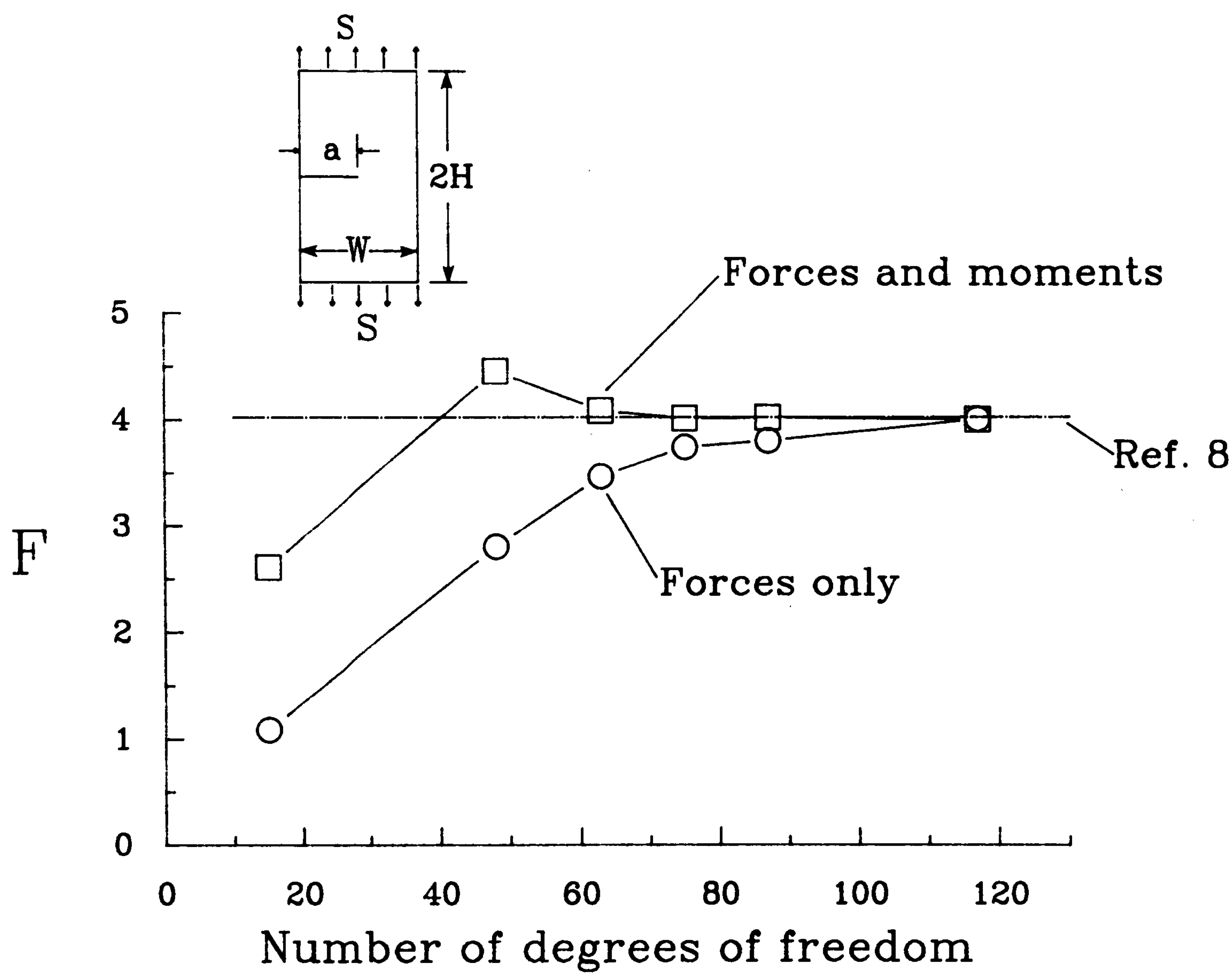

Figure 4. Comparison of convergence of solution with forces only and with forces and moments for single edge crack. $H/W = 2.0$, $a/W = 0.6$.

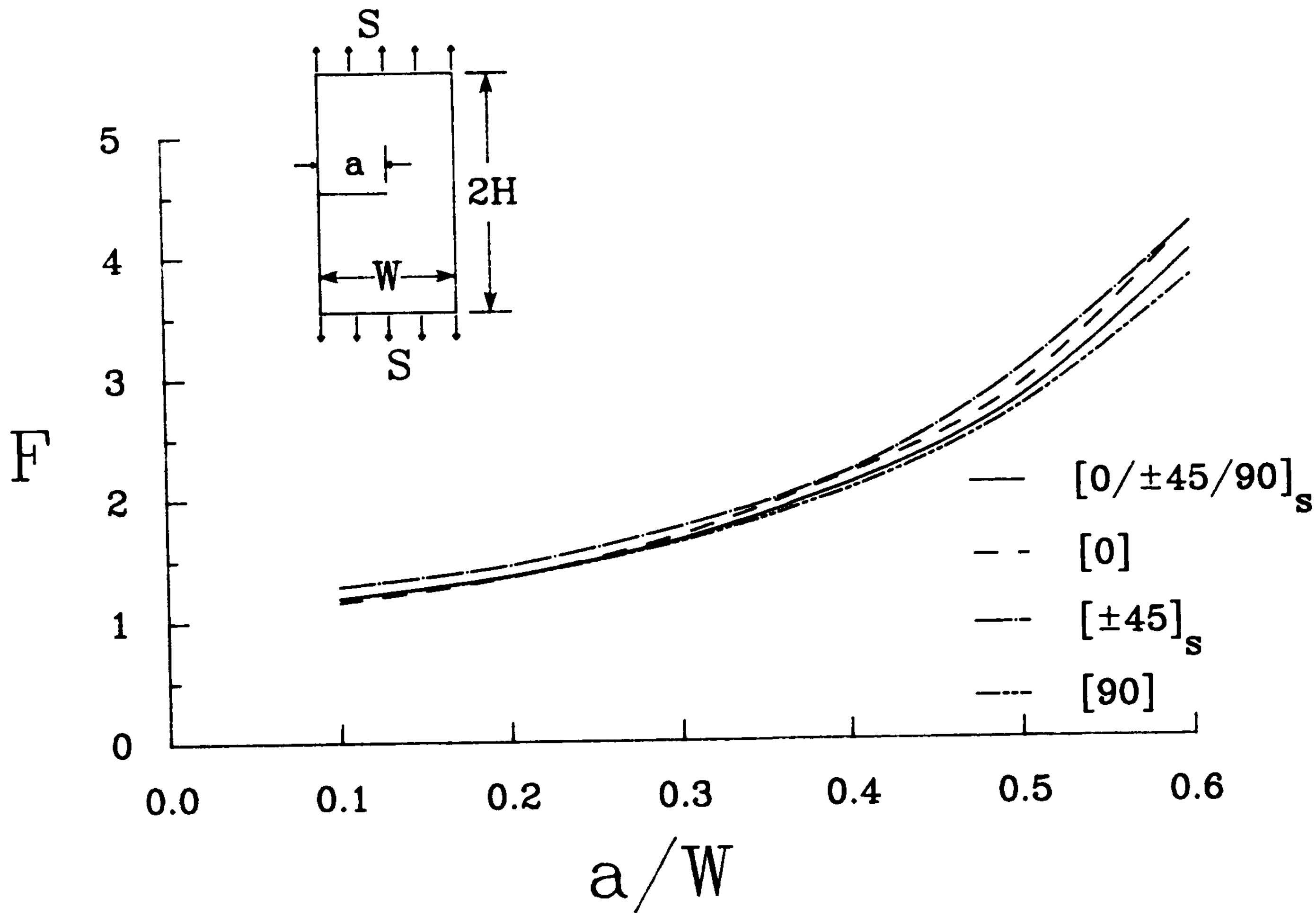

Figure 5. Stress-intensity corrections factors for single edge crack. H/W = 2.0.

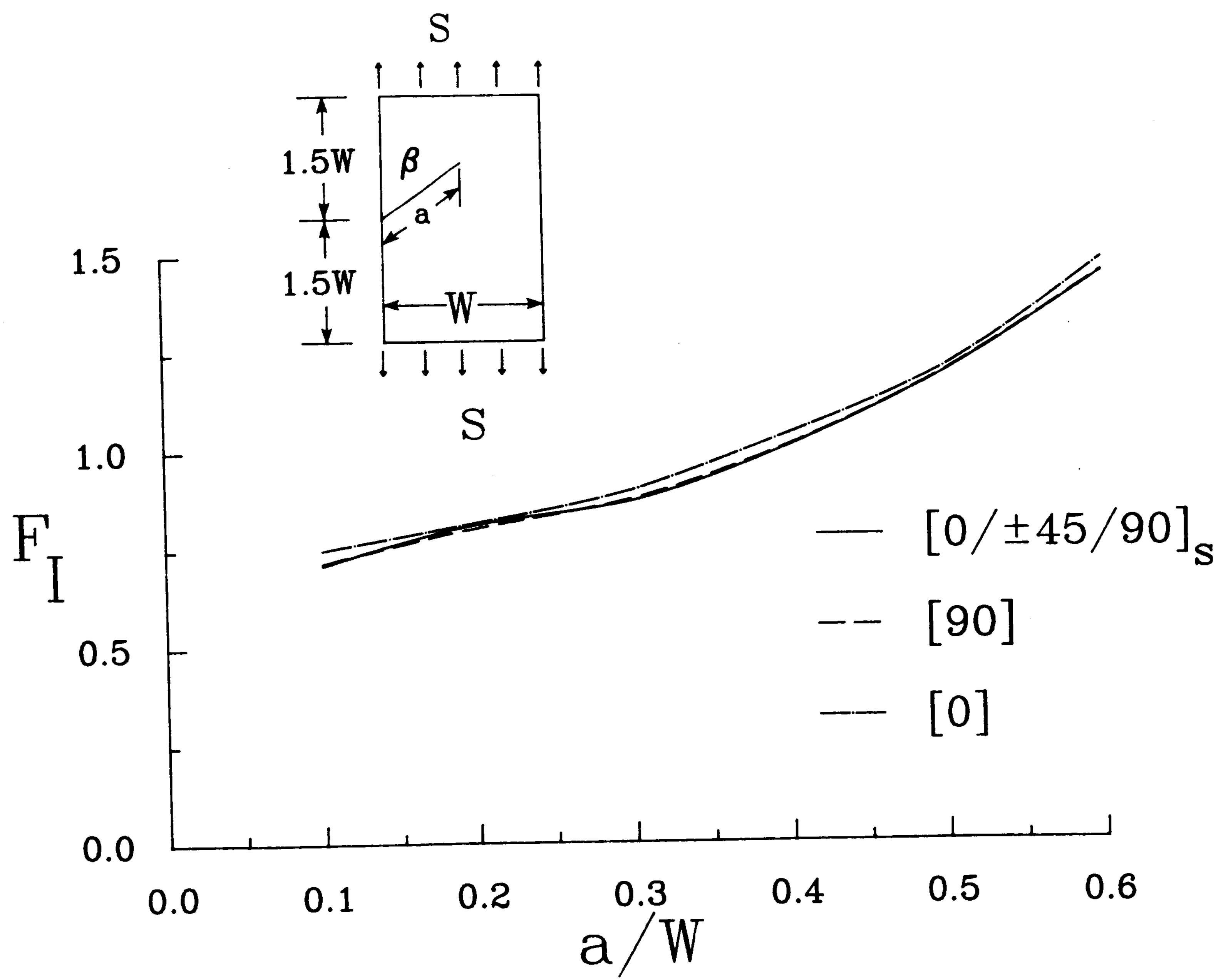

Figure 6. Mode I stress-intensity correction factors for inclined edge crack. $\beta = 45^\circ$, $H/W = 2.0$.

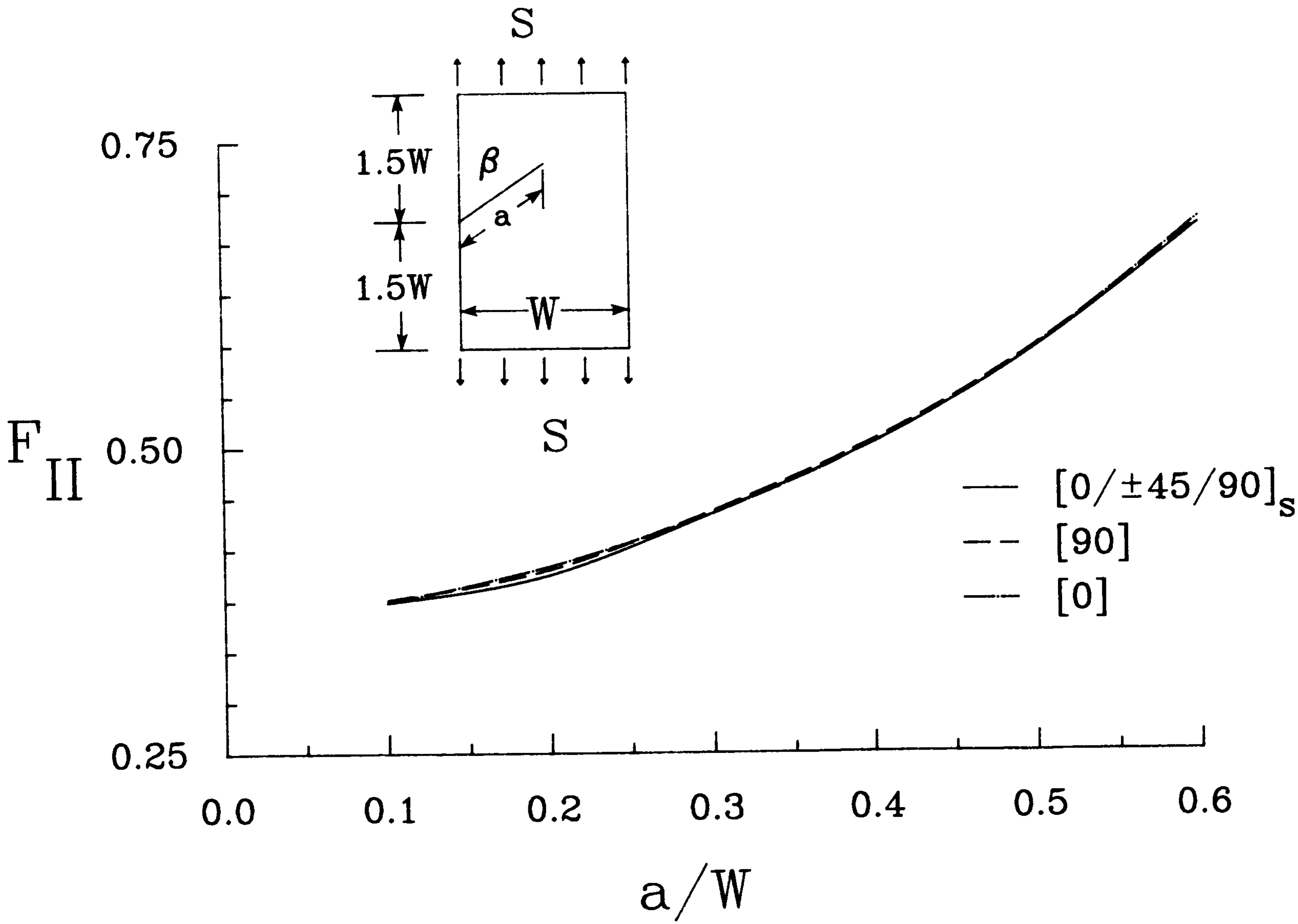

Figure 7. Mode II stress-intensity correction factors for inclined edge crack. $\beta = 45^{\circ}$, $H/W = 2.0$.

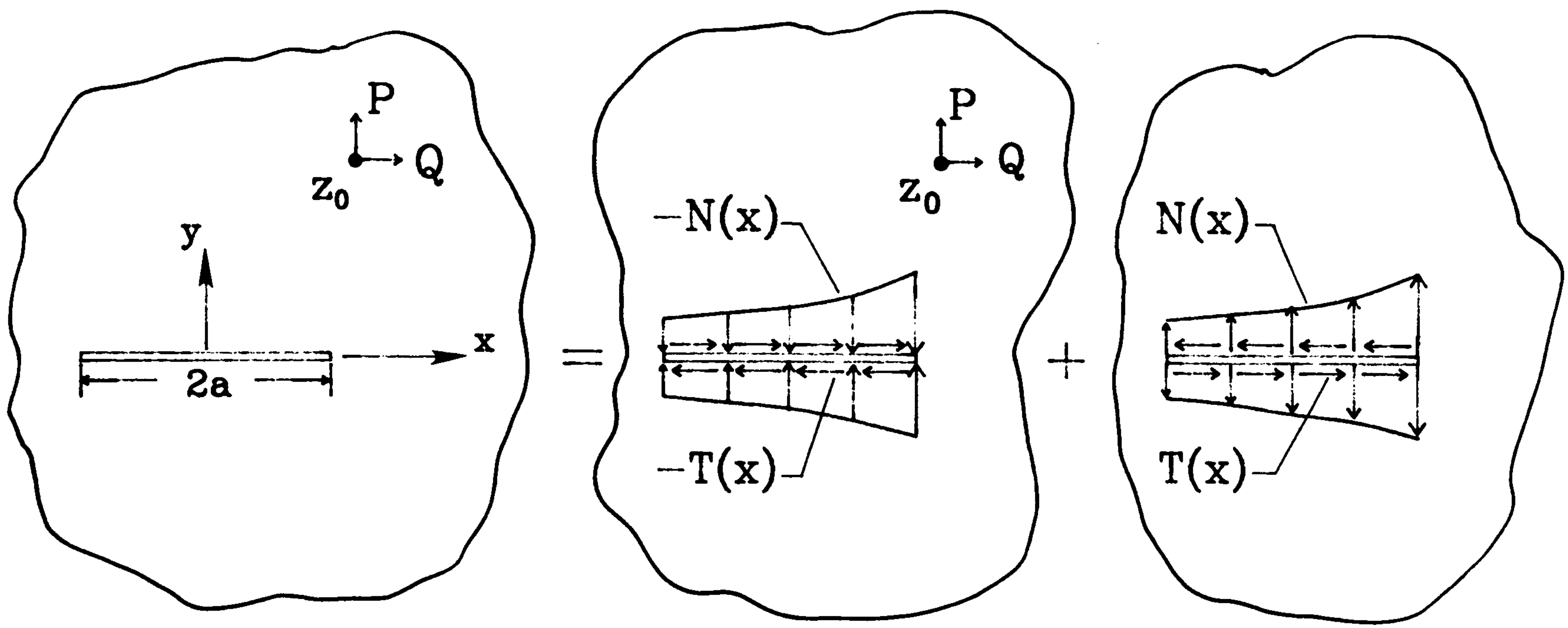

Figure 8. Superposition of stress functions for points loads in orthotropic sheet.

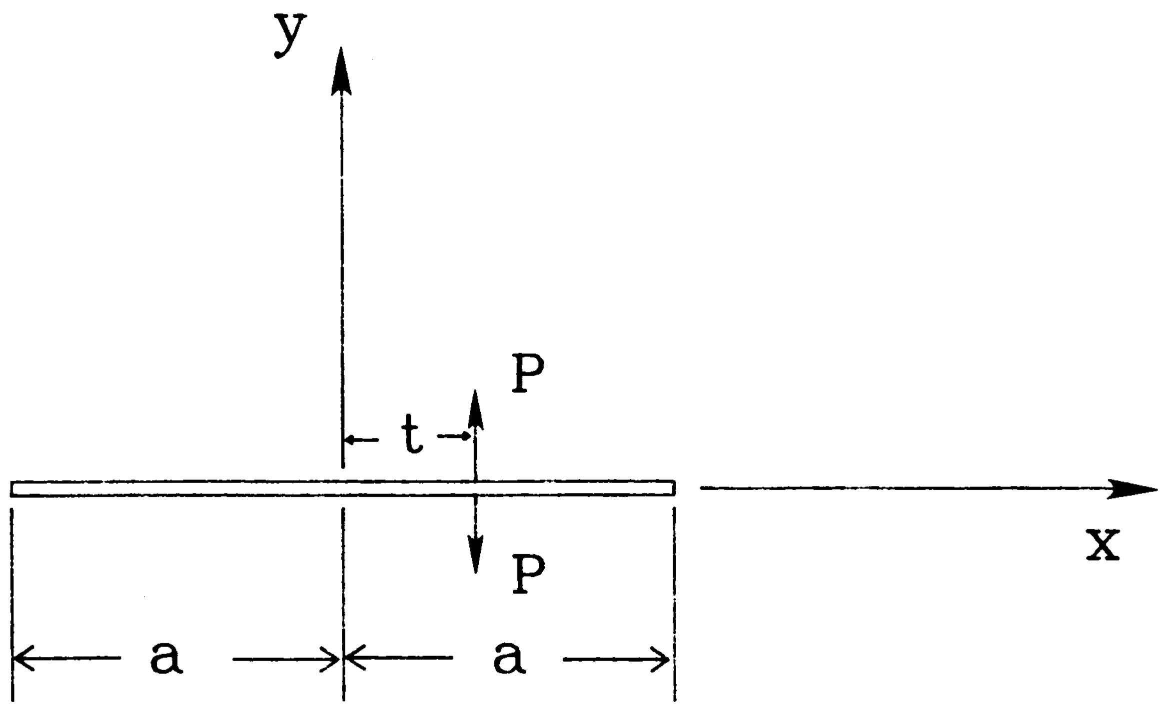

Figure 9. Point loading on crack face.

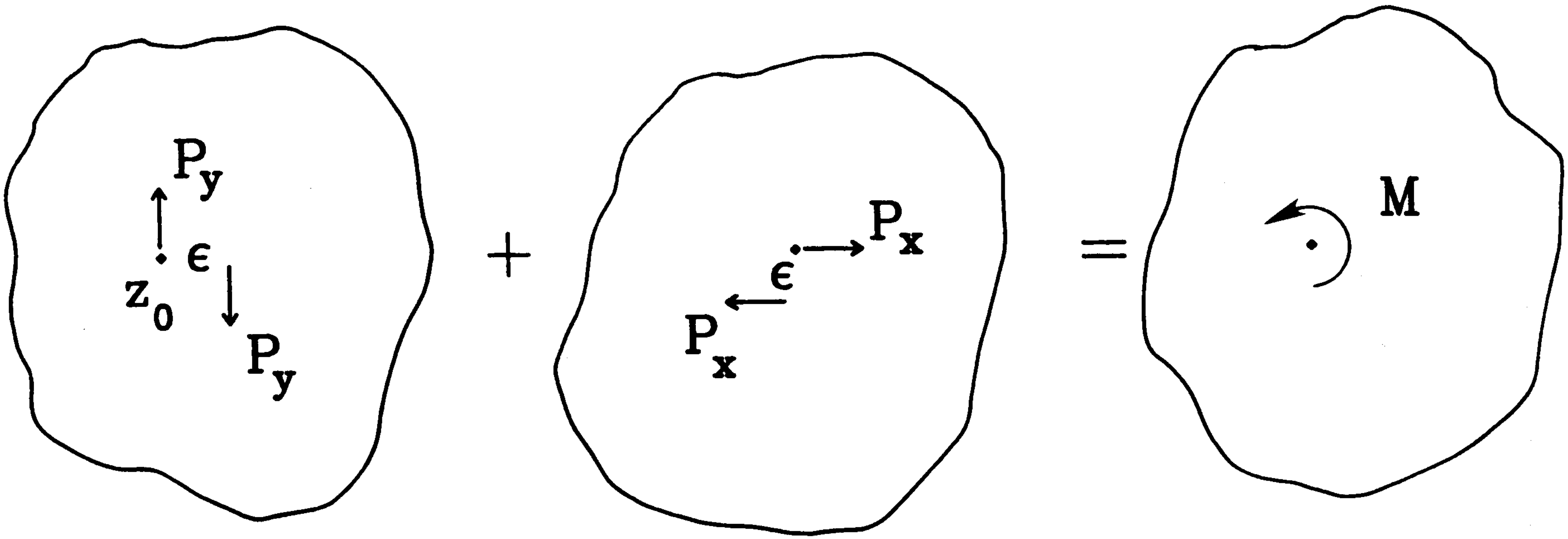

Figure 10. Superposition of stress functions for moment in orthotropic sheet.

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16. Abstract

In this paper, the Boundary Force Method (BFM), a form of an indirect boundary element method, is used to analyze composite laminates with cracks. The BFM uses the orthotropic elasticity solution for a concentrated horizontal and vertical force and a moment applied at a point in a cracked, infinite sheet as the fundamental solution. The necessary stress functions for this fundamental solution were formulated using the complex variable theory of orthotropic elasticity. The current method is an improvement over a previous method that used only forces and no moment. The improved method was verified by comparing it to accepted solutions for a finite-width, center-crack specimen subjected to uniaxial tension. Four graphite/epoxy laminates were used: $[0/\pm45/90]_s$, [0], $[\pm45]_s$, and $[\pm30]_s$. The BFM results agreed well with accepted solutions. Convergence studies showed that with the addition of the moment in the fundamental solution, the number of boundary elements required for a converged solution was significantly reduced. Parametric studies were done for two configurations for which no orthotropic solutions are currently available: a single edge crack and an inclined single edge crack.

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